

Dark energy and moduli stabilization of extra dimensions in $\mathbb{M}^{1+3} \times \mathbb{T}^2$ spacetime

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ABSTRACT: Recently, it was found by Greene and Levin that the Casimir energy of certain combinations of massless and massive fields in space with extra dimensions play a crucial role in the accelerated expansion of the late-time universe and therefore it could serve as a candidate for the dark energy. It also provides a mechanism in stabilizing the volume moduli of extra dimensions. However, the shape moduli of the extra dimensions were never taken into account in the previous work. We therefore study the stabilization mechanism for both volume and shape moduli due to the Casimir energy in $\mathbb{M}^{1+3} \times \mathbb{T}^2$. The result of our study shows that the previously known local minimum is a saddle point. It is unstable to the perturbations in the direction of the shape moduli. The new stable local minima stabilizes all the moduli and drives the accelerating expansion of the universe. The cosmological dynamics both in the bulk and the radion pictures are derived and simulated. The equations of state for the Casimir energy in a general torus are derived. Shear viscosity in extra dimensions induced by the Casimir density in the late times is identified and calculated, it is found to be proportional to the Hubble constant.

KEYWORDS: Large Extra Dimensions, Cosmology of Theories beyond the SM.

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1. Introduction

According to the latest data on Type Ia Supernovae [1] and Cosmic Microwave Background Radiation (CMBR) [2], it is strongly believed that the universe consists of a sort of vacuum energy, namely dark energy, which contributes the accelerated expansion in three-dimensional space. Unfortunately, the exact form of the dark energy has not yet been uncovered until now. The prominent candidates for dark energy are the cosmological constant, and models of scalar fields, such as the quintessence and moduli fields.

In the standard cosmological model where the acceleration of the universe is taken into account by a positive cosmological constant term, dark energy contributes largely, more than 70 % of the total density of the universe [2]. This number (roughly 10^{-11} eV^4) seems arbitrarily small and the known mechanisms, such as the popular TeV-scale supersymmetry (SUSY) breaking scenario or any top-down high-scale particle physics mechanisms, fail to produce it.

In recent years, theories with large extra dimensions have received an explosion of interests as they provide new solution to the hierarchy problem. Recently, it was found that Casimir energy of massless and massive fields embedded in higher-dimensional space-time could play a crucial role of dark energy with additional significant properties [3, 4].

The Casimir energy not only drives the expansion of universe acceleratedly, but also stabilizes the volume moduli of extra dimensions. However, the shape moduli, τ_1, τ_2 , were not included in the work of Greene and Levin. In this work we therefore take into account these moduli in the cosmological dynamics by assuming that the extra dimensions are \mathbb{T}^2 . The phenomenological implications of nontrivial shape moduli were pointed out in [5–7]. Shape moduli can have dramatic effects on the Kaluza-Klein spectrum, for example, they can induce level-crossings and varying mass gaps. They can also help to eliminate light KK states. It should be interesting to investigate the role of shape moduli in cosmology.

Our work employed the calculation of Casimir energy in the non-trivial space $\mathbb{M}^4 \times \mathbb{T}^2$. The Casimir energy is the vacuum energy contributed from the quantum fluctuation of fields which satisfy certain boundary conditions. In fact, the Casimir energy in various spaces including a distorted torus was studied in earlier works [8–10, 4]. The standard approach for determining the Casimir energy is the zeta function regularization [11].

Our result shows that the minimum of potential in the previous work [3] ($\tau_1 = 0, \tau_2 = 1$) was the unstable local minimum while the true local minimum locates at specific points in the moduli space, $\tau_1 = \pm 1/2, \tau_2 = \sqrt{3}/2$, confirming the result of ref. [4]. At this local minimum the potential stabilizes all moduli and also sources the accelerated expansion of the four dimensional universe.

This paper is organized as follows. In section 2 we review cosmological dynamics on $\mathbb{M}^{1+n} \times \mathbb{T}^p$ spacetime. In section 3 we present the mathematical calculation to determine the Casimir energy of massive and massless fields in the spacetime with toroidally compactified extra dimensions. Then we go on to construct effective potential contributed by Casimir energy of massive and massless field in $\mathbb{M}^{1+3} \times \mathbb{T}^2$ spacetime in section 4. The numerical evidences of the stability of moduli space are presented in section 5. In section 6 we present our conclusions.

2. Cosmological dynamics in $\mathbb{M}^{1+n} \times \mathbb{T}^p$

Our study of cosmological dynamics is based upon the application of Einstein’s general relativity on the product space $\mathbb{M}^{1+n} \times \mathbb{T}^p$, between a $(1+n)$ -dimensional spacetime and a p -dimensional toroidally-compactified space. As a whole, the total number of spatial dimensions is $d = n + p$. We assume the cosmological ansatz

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + h_{ij}(x)dy^i dy^j, \tag{2.1}$$

where the metric h_{ij} represent the p -dimensional compact space with $i, j = 1, \dots, p$ and $g_{\mu\nu}$ for the $(1+n)$ -dimensional noncompact spacetime with $\mu, \nu = 0, \dots, n$. Let’s assume also that the metric only depends on the noncompact coordinates x^μ . The compact coordinates are $0 \leq y^i \leq 2\pi$.

In this paper, we focus our effort on the cosmological dynamics of a 4-dimensional spacetime with two extra dimensions ($n = 3$ and $p = 2$). The metric of two-dimensional torus \mathbb{T}^2 takes the form

$$(h_{ij}) = \frac{b^2}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \tag{2.2}$$

where $\tau = \tau_1 + i\tau_2$ is the complex structure (or shape moduli) and b^2 is the Kähler structure (or volume moduli). In cosmology, it is customary to write $g_{\mu\nu} = a^2(t)\eta_{\mu\nu}$ and $h_{ij} = h_{ij}(t)$. In the next sections, we will assume that Casimir energy in compact direction, $\rho_{(d+1)D}$, plays the roles of the dominant energy content in the universe. By using Einstein equations in $(1+5)$ -dimensional spacetime, we obtain the following equations governing the cosmological dynamics:

$$3H_a^2 + H_b^2 + 6H_a H_b - \frac{1}{4\tau_2^2}(\dot{\tau}_1^2 + \dot{\tau}_2^2) = 8\pi G \rho_{6D}, \quad (2.3)$$

$$\begin{aligned} \dot{H}_a + 3H_a^2 + 2H_a H_b = \frac{8\pi G}{4} \left\{ 2\rho_{6D} + \left[1 - \left(\frac{\tau_1}{\tau_2} \right)^2 \right] b \partial_b \rho_{6D} \right. \\ \left. - 2\tau_1 \partial_{\tau_1} \rho_{6D} + \frac{2\tau_1^2}{\tau_2} \partial_{\tau_2} \rho_{6D} \right\}, \end{aligned} \quad (2.4)$$

$$\begin{aligned} \dot{H}_b + 2H_b^2 + 3H_a H_b = -\frac{8\pi G}{4} \left\{ -2\rho_{6D} + \left[1 - \left(\frac{\tau_1}{\tau_2} \right)^2 \right] b \partial_b \rho_{6D} \right. \\ \left. - 2\tau_1 \partial_{\tau_1} \rho_{6D} + \frac{2\tau_1^2}{\tau_2} \partial_{\tau_2} \rho_{6D} \right\}, \end{aligned} \quad (2.5)$$

$$\ddot{\tau}_1 + \left(3H_a + 2H_b - 2\frac{\dot{\tau}_2}{\tau_2} \right) \dot{\tau}_1 = -16\pi G \tau_2^2 \left\{ \frac{b\tau_1}{2\tau_2^2} \partial_b \rho_{6D} + 2\partial_{\tau_1} \rho_{6D} - \frac{\tau_1}{\tau_2} \partial_{\tau_2} \rho_{6D} \right\}, \quad (2.6)$$

$$\begin{aligned} \frac{\ddot{\tau}_2}{\tau_2} + \frac{\dot{\tau}_1^2 - \dot{\tau}_2^2}{\tau_2^2} + 3H_a \frac{\dot{b}}{\tau_2} + 2H_b \frac{\dot{\tau}_2}{\tau_2} = 8\pi G \left\{ \frac{b\tau_1^2}{\tau_2^2} \partial_b \rho_{6D} + 2\tau_1 \partial_{\tau_1} \rho_{6D} \right. \\ \left. - 2\tau_2 \left[1 + \left(\frac{\tau_1}{\tau_2} \right)^2 \right] \partial_{\tau_2} \rho_{6D} \right\}. \end{aligned} \quad (2.7)$$

where G is the 6-D gravitational constant. We have defined the Hubble constants $H_a = \dot{a}/a$ and $H_b = \dot{b}/b$, where a dotted quantity represents the corresponding time derivative and ρ_{6D} is the casimir energy density in six dimensional spacetime.

2.1 Dynamics in the radion picture

Equations of motion (2.3)–(2.7) can be obtained by varying the $d+1$ -dimensional Einstein-Hilbert action:

$$S = \int d^{1+n} x d^p y \sqrt{-gh} \left\{ \frac{M_*^{d-1}}{16\pi} \mathcal{R}_{(1+d)} - \rho_{(1+d)D}(h^{ij}) \right\}, \quad (2.8)$$

with $n = 3$ and $p = 2$, where $\rho_{(1+d)D}(h^{ij})$, $\mathcal{R}_{(1+d)}$ and M_* are the Casimir energy density, Ricci scalar and the Planck mass in $(1+d)$ -dimensional spacetime respectively. For later purpose, it is useful to perform KK-dimensional reduction of the above action from $(1+d)$ to $(1+n)$ -dimensional spacetime and Weyl rescaling $g_{\mu\nu_E} = \Omega^{\frac{2}{n-1}} g_{\mu\nu}$; $\Omega = M_*^{d-1} V_p / m_{pl}^{n-1}$, the action takes the form

$$S = \int d^{1+n} x \sqrt{-g_E} \left\{ \frac{m_{pl}^{n-1}}{16\pi} \left[\mathcal{R}_E + g_E^{\mu\nu} \left(\frac{1}{1-n} \nabla_\mu \ln \sqrt{h} \nabla_\nu \ln \sqrt{h} + \frac{1}{4} \nabla_\mu h^{ij} \nabla_\nu h_{ij} \right) \right] - U(h^{ij}) \right\}. \quad (2.9)$$

Note that the subscript E denotes the Einstein frame variables. Here, $V_p = \int d^p y \sqrt{h} = (2\pi b)^p \equiv l^p$ is the (invariant) volume of extra dimensions, m_{pl} and $U(h^{ij}) = \Omega^{\frac{1+n}{1-n}} V_p \rho_{(1+n)D}(h^{ij}) = \Omega^{\frac{1+n}{1-n}} \rho_{(1+n)D}(h^{ij})$ are the Planck mass and the effective potential in $1+n$ -dimensional spacetime respectively. We can also take $\rho_{(1+n)D}(h^{ij})$ to be the Casimir energy density in $(1+n)$ -dimensional spacetime.

Since we are interested in the $n=3, p=2$ case, by using the metric of two-dimensional torus defined in eq. (2.2), the action in eq. (2.9) can be written as

$$S = \int d^4 x \sqrt{-g_E} \left\{ \frac{m_{pl}^2}{16\pi} \left[\mathcal{R}_E - \frac{1}{2} g_E^{\mu\nu} (\nabla_\mu \psi \nabla_\nu \psi + e^{-2\phi_2} \nabla_\mu \phi_1 \nabla_\nu \phi_1 + \nabla_\mu \phi_2 \nabla_\nu \phi_2) \right] - U(\psi, \phi_1, \phi_2) \right\}, \quad (2.10)$$

where $\psi \equiv 2\sqrt{2} \ln b$, $\phi_1 \equiv \tau_1$, and $\phi_2 \equiv \ln \tau_2$. Such action gives rise to the following set of equations:

$$6H_E^2 - \frac{1}{2} (\dot{\psi}^2 + e^{-2\phi_2} \dot{\phi}_1^2 + \dot{\phi}_2^2) = \frac{16\pi}{m_{pl}^2} U, \quad (2.11)$$

$$\ddot{\psi} + 3H_E \dot{\psi} = -\frac{16\pi}{m_{pl}^2} \frac{\partial U}{\partial \psi}, \quad (2.12)$$

$$\ddot{\phi}_1 + 3H_E \dot{\phi}_1 - 2\dot{\phi}_1 \dot{\phi}_2 = -\frac{16\pi}{m_{pl}^2} e^{2\phi_2} \frac{\partial U}{\partial \phi_1}, \quad (2.13)$$

$$\ddot{\phi}_2 + 3H_E \dot{\phi}_2 + e^{-2\phi_2} \dot{\phi}_1^2 = -\frac{16\pi}{m_{pl}^2} \frac{\partial U}{\partial \phi_2}, \quad (2.14)$$

and

$$4\dot{H}_E + (\dot{\psi}^2 + e^{-2\phi_2} \dot{\phi}_1^2 + \dot{\phi}_2^2) = 0. \quad (2.15)$$

Note that $H_E = (da_E/dt_E)/a_E$ is the Hubble constant in the Einstein's frame.

3. Casimir energy in $\mathbb{M}^{1+n} \times \mathbb{T}^p$

In this section, we will undergo the mathematical formulation to determine the Casimir energy, \hat{E}_{cas} , associated with a scalar field of mass M in a $\mathbb{M}^{1+n} \times \mathbb{T}^p$ space. The fermionic degree of freedom will contribute to the Casimir energy with the same expression except for an extra minus sign. We then focus on the result from our phenomenological study ($n=3, p=2$).

3.1 Casimir-energy calculation

Let $V_n = L^n$ be the spatial volume of non-compact spacetime, and $V_p = l^p$ be the volume of compact space. If we assume $L \gg l$, the zero-point energy of scalar fields in $\mathbb{M}^{1+n} \times \mathbb{T}^p$ can be evaluated by

$$\hat{E}_{\text{cas}} = \frac{1}{2} \left(\frac{L}{2\pi} \right)^n \sum_{n_i, n_j} \int_{-\infty}^{+\infty} d^n k \sqrt{\delta^{ab} k_a k_b + h^{ij} n_i n_j + M^2}, \quad (3.1)$$

where k_a ; $a=1, \dots, n$ is the momentum in each non-compact spatial direction, $n_i \in \mathbb{Z}$; $i=1, \dots, p$ is the momentum number in each compact direction.

Using the property of integration in appendix A and changing variable of integration as $v = k^2/(h^{ij}n_i n_j + M^2)$, we can express the Casimir energy as

$$\widehat{E}_{\text{cas}} = \frac{1}{2} \left(\frac{L}{2\pi} \right)^n \frac{\pi^{n/2}}{\Gamma(n/2)} \sum_{n_i, n_j} (h^{ij}n_i n_j + M^2)^{\frac{n+1}{2}} \int_0^\infty dv v^{\frac{n-2}{2}} \sqrt{1+v}. \quad (3.2)$$

We can convert the integral into the Gamma function by using the formulae in appendix A; as a consequence, we obtain the Casimir energy in a simple form

$$\widehat{E}_{\text{cas}} = \frac{1}{2} \left(\frac{2\pi}{L} \right)^{1+2s} \frac{\Gamma(s)}{\pi^{\frac{1+2s}{2}} \Gamma(-\frac{1}{2})} \sum_{n_i, n_j} (h^{ij}n_i n_j + M^2)^{-s}; \quad s = -\frac{d-p+1}{2}. \quad (3.3)$$

In our case, the compact space is \mathbb{T}^2 and h^{ij} is the inverse metric from eq. (2.2); therefore, our next task is to regularize the infinite summation in the eq. (3.3)

$$F \left(s; \frac{|\tau|^2}{b^2\tau_2}, -\frac{2\tau_1}{b^2\tau_2}, \frac{1}{b^2\tau_2}; M^2 \right) = \sum_{n_1, n_2} \left(\frac{|\tau|^2}{b^2\tau_2} n_1^2 - \frac{2\tau_1}{b^2\tau_2} n_1 n_2 + \frac{1}{b^2\tau_2} n_2^2 + M^2 \right)^{-s}, \quad (3.4)$$

which is known as extended Chowla-Selberg zeta function [9]. It is worth noting that $V_p = l^2 = (2\pi b)^2$ in this case.

After a few steps of analytic manipulation by using Poisson resummation and property of the modified Bessel function, we obtain

$$\begin{aligned} F \left(s; \frac{|\tau|^2}{b^2\tau_2}, -\frac{2\tau_1}{b^2\tau_2}, \frac{1}{b^2\tau_2}; M^2 \right) &= b^{2s} \left\{ 2\tau_2^s \zeta_{\text{EH}}(s; \tau_2 b^2 M^2) \right. \\ &\quad + 2\sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \tau_2^{1-s} \zeta_{\text{EH}} \left(s - 1/2; \frac{b^2 M^2}{\tau_2} \right) \\ &\quad \left. + \sum_{m, k=1}^{\infty} \frac{8\pi^s}{\Gamma(s)} \sqrt{\tau_2} k^{s-\frac{1}{2}} \frac{\cos(2\pi\tau_1 m k)}{\left(\sqrt{m^2 + \frac{b^2 M^2}{\tau_2}} \right)^{s-\frac{1}{2}}} K_{s-\frac{1}{2}} \left(2\pi\tau_2 k \sqrt{m^2 + \frac{b^2 M^2}{\tau_2}} \right) \right\}, \end{aligned} \quad (3.5)$$

where the Epstein-Hurwitz zeta function $\zeta_{\text{EH}}(s; q)$ is expressed as

$$\begin{aligned} \zeta_{\text{EH}}(s; q) &= \frac{1}{2} \sum'_{n \in \mathbb{Z}} (n^2 + q)^{-s} \\ &= -\frac{q^{-s}}{2} + \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{2\Gamma(s)} q^{-s+\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{2\pi^s q^{-s/2+1/4}}{\Gamma(s)} n^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi n \sqrt{q}), \end{aligned} \quad (3.6)$$

where the prime at the first sum indicates that the term $n = 0$ is excluded. A similar expression which manifests the periodicity of the Casimir energy with respect to τ_1 is also given in ref. [12].

The expression serves as an analytic continuation of the Casimir energy where s is extended from positive to negative values. Inserting eq. (3.5) into eq. (3.3) and eliminating the infinite terms due to the pole of $\Gamma(s = -2)$ and $\Gamma(s - 1 = -3)$ in this case, we conveniently reached the regularized Casimir energy. The dropped divergent terms correspond

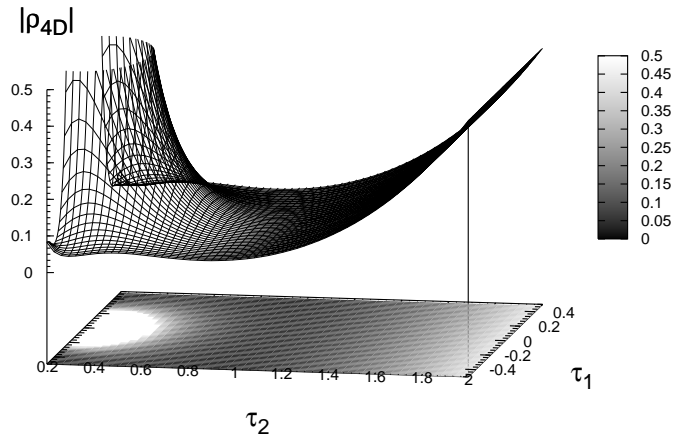


Figure 1: The magnitude of the Casimir energy density, $|\rho_{4D}|$, in four dimension per degree of freedom for $M = 5, b = 0.133$.

to the constant total energy and the constant energy density in the bulk. Both of them do not depend on any parameters of the torus and therefore can be safely eliminated from the physically relevant Casimir effects by renormalization. The final regulated Casimir energy density $\rho(h^{ij})$ in (1 + 3)-dimensional spacetime can then be expressed as

$$\begin{aligned}
 \rho_{4D}(b^2, \tau_1, \tau_2) &= \frac{\widehat{E}_{cas}}{V_m} \\
 &= -(4\pi^2 b^2)^s \left\{ 2\tau_2^s (\tau_2 b^2 M^2)^{-\frac{s}{2} + \frac{1}{4}} \sum_{k=1}^{\infty} k^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi k b M \sqrt{\tau_2}) \right. \\
 &\quad + 2\tau_2^{1-s} \left(\frac{b^2 M^2}{\tau_2} \right)^{-\frac{s}{2} + \frac{1}{2}} \sum_{k=1}^{\infty} k^{s-1} K_{s-1} \left(\frac{2\pi k b M}{\sqrt{\tau_2}} \right) \\
 &\quad \left. + 4\sqrt{\tau_2} \sum_{k,m=1}^{\infty} k^{s-\frac{1}{2}} \frac{\cos(2\pi\tau_1 k m)}{\left(\sqrt{m^2 + \frac{b^2 M^2}{\tau_2}} \right)^{s-\frac{1}{2}}} K_{s-\frac{1}{2}} \left(2\pi k \tau_2 \sqrt{m^2 + \frac{b^2 M^2}{\tau_2}} \right) \right\}. \quad (3.7)
 \end{aligned}$$

In the case of massless scalar fields ($M = 0$), the Casimir energy density becomes

$$\begin{aligned}
 \rho_{4D}(b^2, \tau_1, \tau_2) &= -(4\pi^2 b^2)^s \left\{ \tau_2^s \pi^{s-\frac{1}{2}} \Gamma\left(\frac{1}{2} - s\right) \zeta(1 - 2s) + \tau_2^{1-s} \pi^{s-1} \Gamma(1 - s) \zeta(2 - 2s) \right. \\
 &\quad \left. + 4\sqrt{\tau_2} \sum_{m,k=1}^{\infty} \left(\frac{k}{m} \right)^{s-\frac{1}{2}} \cos(2\pi m k \tau_1) K_{s-\frac{1}{2}}(2\pi m k \tau_2) \right\}. \quad (3.8)
 \end{aligned}$$

The Casimir density in (1+3+2) dimensions is given by $\rho_{6D} = \rho_{4D}/(2\pi b)^2$.

As it is pointed out in the work of Ponton and Poppitz [4]. Since the symmetry $\tau \rightarrow -1/\tau, \tau \rightarrow \tau + 1$ of the torus is preserved in the Casimir energy expression, it is sufficient

to consider only the fundamental region where $\tau \geq 1, -1/2 \leq \tau_1 \leq 1/2$ of the shape moduli space. In the fundamental region, there are two minima and one saddle point of the magnitude $|\rho|$ of the Casimir energy density. The saddle point locates at $\tau_1 = 0, \tau_2 = 1$ and the two minima locate at $\tau_1 = \pm 1/2, \tau_2 = \sqrt{3}/2$. This is shown in figure 1.

3.2 Analysis for small bM

In the limit of $bM \ll 1$, we recalculate the Casimir energy by performing the binomial expansion with respect to small bM before regularization, and keep only the leading-order terms. It can be demonstrated that the process of regularizing each term after performing binomial expansion is NOT equivalent to the process of regularizing the whole expression at once if $s = -2$ is set beforehand. When we set $s = (1-d)/2 = -2$, the binomial expansion of eq. (3.4) gives only three terms with orders of $(bM)^0, (bM)^2$, and $(bM)^4$, whereas the regularization of the full expression before setting $s = -2$ as in eq. (3.5), which gives eq. (3.7) as a result, generically leads to an infinite series of bM , even after setting $s = -2$ in the final expression.

Without setting $s = -2$ before regularization, the precise dependence of the coefficients of the bM -binomial expansion to the moduli parameters τ_1, τ_2 will be determined. The small bM expansion is obtained subsequently.

We begin by replacing h^{ij} with the form of the inverse metric of \mathbb{T}^2 in eq. (3.3) and using Mellin transform (see appendix A)

$$\begin{aligned} \widehat{E}_{\text{cas}} &= \frac{1}{2} \left(\frac{2\pi}{L} \right)^{1+2s} \frac{\Gamma(s)}{\pi^{\frac{1+2s}{2}} \Gamma(-\frac{1}{2})} \sum_{n_1, n_2 \in \mathbb{Z}} \int_0^\infty dt t^{s-1} e^{-\{\frac{1}{\tau_2 b^2} (|\tau|^2 n_1^2 - 2\tau_1 n_1 n_2 + n_2^2) + M^2\}t} \quad (3.9) \\ &= \frac{1}{2} \left(\frac{2\pi}{L} \right)^{1+2s} \frac{\Gamma(s)}{\pi^{\frac{1+2s}{2}} \Gamma(-\frac{1}{2})} \sum_{n_1, n_2 \in \mathbb{Z}} \int_0^\infty dv v^{s-1} e^{-\{\frac{1}{\tau_2} (|\tau|^2 n_1^2 - 2\tau_1 n_1 n_2 + n_2^2) + (bM)^2\}v} \\ &= \frac{1}{2} \left(\frac{2\pi}{L} \right)^{1+2s} \frac{b^{2s}}{\pi^{\frac{1+2s}{2}} \Gamma(-\frac{1}{2})} \sum_{j=0}^\infty \frac{(-1)^j}{j!} (bM)^{2j} \Gamma(s+j) \sum_{n_1, n_2 \in \mathbb{Z}} \left(\frac{|\tau|^2}{\tau_2} n_1^2 - 2\frac{\tau_1}{\tau_2} n_1 n_2 + \frac{1}{\tau_2} n_2^2 \right)^{-(s+j)}, \end{aligned}$$

where the second line is obtained by changing the dummy variable $v = t/b^2$, and the final line is obtained by expanding the Taylor series for $e^{-(bM)^2}$. We can determine the double summation in eq. (3.9) by using the result in eq. (3.4), (3.5); as a consequence, the Casimir energy density in five spatial dimensions takes the form,

$$\begin{aligned} \rho_{6D}(b^2, \tau_1, \tau_2) &= -(4\pi^2 b^2)^{s-1} \sum_{j=0}^\infty \frac{(-1)^j}{j!} (bM)^{2j} \\ &\times \left\{ 4\pi^j \sqrt{\tau_2} \sum_{m,k=1}^\infty \left(\frac{k}{m} \right)^{s+j-\frac{1}{2}} \cos(2\pi m k \tau_1) K_{s+j-\frac{1}{2}}(2\pi m k \tau_2) \right. \\ &\left. + \pi^{s+2j-\frac{1}{2}} \tau_2^{s+j} \Gamma\left(\frac{1}{2}-s-j\right) \zeta(1-2s-2j) + \pi^{s+2j-1} \tau_2^{1-s-j} \Gamma(1-s-j) \zeta(2-2s-2j) \right\}. \end{aligned}$$

In the limit $bM \ll 1$ for $s = -2$, the Casimir energy density then becomes

$$\rho_{6D}(b^2, \tau_1, \tau_2) \simeq -\frac{1}{(4\pi^2 b^2)^3} \left\{ C_1 - C_2 (bM)^2 + C_3 (bM)^4 \right\} \quad (3.10)$$

where

$$\begin{aligned}
 C_1 &\equiv \pi^{-\frac{5}{2}}\tau_2^{-2}\Gamma\left(\frac{5}{2}\right)\zeta(5) + \pi^{-3}\tau_2^3\Gamma(3)\zeta(6) + 4\sqrt{\tau_2} \sum_{m,k=1}^{\infty} \left(\frac{m}{k}\right)^{\frac{5}{2}} \cos(2\pi mk\tau_1) K_{-5/2}(2\pi mk\tau_2), \\
 C_2 &\equiv \pi^{-\frac{1}{2}}\tau_2^{-1}\Gamma\left(\frac{3}{2}\right)\zeta(3) + \pi^{-1}\tau_2^2\Gamma(2)\zeta(4) + 4\pi\sqrt{\tau_2} \sum_{m,k=1}^{\infty} \left(\frac{m}{k}\right)^{\frac{3}{2}} \cos(2\pi mk\tau_1) K_{-3/2}(2\pi mk\tau_2), \\
 C_3 &\equiv \frac{\pi}{2}\tau_2\Gamma(1)\zeta(2) + 2\pi^2\sqrt{\tau_2} \sum_{m,k=1}^{\infty} \left(\frac{m}{k}\right)^{\frac{1}{2}} \cos(2\pi mk\tau_1) K_{-1/2}(2\pi mk\tau_2). \tag{3.11}
 \end{aligned}$$

In the next section, the total Casimir density for small bM and the full expression will be numerically compared. The true minimum of the potential, induced from the Casimir energy density located at a point $(\tau_1, \tau_2) = (\pm 1/2, \sqrt{3}/2)$, appears only when the full expression is evaluated.

4. Particle spectrum and effective potential for moduli fields

It is demonstrated in ref. [4] and ref. [3] that a careful mixing of massless and massive, bosonic and fermionic degrees of freedom of the bulk fields can lead to a Casimir energy density with local minimum with respect to the scale factor, b , of the compact extra dimensions. In the torus case with the shape moduli τ_1, τ_2 , it can be shown that the true minimum of the mixed Casimir energy density (and thus the potential) locates at $\tau_1 = \pm 1/2, \tau_2 = \sqrt{3}/2$, in contrast to the case of undistorted torus considered in the previous work where the shape moduli are set to $\tau_1 = 0, \tau_2 = 1$.

The simplest model of the bulk fields in our $\mathbb{M}^{1+3} \times \mathbb{T}^2$ space consists of a massless boson, a massless fermion, a massive fermion with mass M , and a massive boson with mass λM . It was found that for the range $0.40 < \lambda < 0.42$ and $M = 5$, the mixed Casimir density has local minimum with respect to the scale factor b , and the moduli τ_1, τ_2 . Since the mass of the boson is different from the mass of the fermion, this is the scenario where SUSY is broken in the bulk if it exists at higher scales. There is no particular reason for why the ratio of the masses of the massive boson and fermion took the specific value in this range. If it has anything to do with SUSY breaking, it is desirable that we are able to establish a SUSY breaking mechanism where this specific ratio of the masses λ could be explained or distinctively selected. From phenomenological point of view, it is desirable that these massless and small-mass bulk fields are sterile neutrinos for they can explain the smallness of neutrino masses in four dimensions. For further details, see ref. [13, 14].

An important issue in mixing bosonic and fermionic degrees of freedom to obtain the total Casimir energy density with a local minimum is the positivity of the energy density. Generally, the value of the total Casimir density at $\tau_1 = \pm 1/2, \tau_2 = \sqrt{3}/2$ is lower than the value at the saddle point $\tau_1 = 0, \tau_2 = 1$, for all range of λ . However, for certain ranges of λ (e.g. $\lambda \lesssim 0.407$), the density becomes negative around the true minimum and therefore violates the positive energy condition. A negative value of the density will not stabilize the dynamics and the size of the torus. We therefore choose the value $\lambda = 0.408$ for our

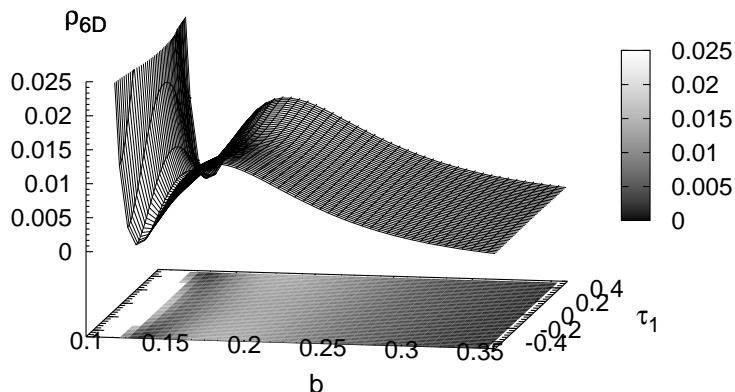


Figure 2: The total Casimir energy density in six dimension for mixture of massless and massive fields for $M = 5$, $\lambda = 0.408$, and $\tau = \sqrt{\tau_1^2 + \tau_2^2}$ is fixed to 1.

simulation of the cosmological dynamics. Figure 2 shows the total Casimir energy density for the spectrum of massless and massive particles mentioned above.

The plot of the total Casimir density in (1+3+2)-dimensional spacetime using the full expression, eq. (3.7), in comparison to the plot from the small bM approximation, eq. (3.10), is given in figure 3. The true minimum at $\tau_1 = \pm 1/2, \tau_2 = \sqrt{3}/2$ only exist in the full expression case. This can be understood considering $b_{\min}M \approx 0.67$ and is somewhat close to 1, resulting in a bad approximation of the expression due to higher powers of bM being neglected. It is therefore required that we use the full expression of the total Casimir energy density in the simulation of the cosmological dynamics.

5. Evidence of stability of the moduli space and cosmological dynamics

By numerically solving the field equations in section 2, the stabilization of the torus and the accelerated expansion of large 4-dimensional spacetime can be demonstrated to occur at the true minimum of the Casimir energy density in the moduli space. The point $\tau_1 = 0, \tau_2 = 1$ is a saddle point and it is an unstable equilibrium of the dynamics.

The rolling of the universe to the true minimum of the Casimir density is illustrated in figure 4–7. When the cosmological dynamics is initiated even within a small vicinity of the saddle point, $\tau_1 = 0, \tau_2 = 1$, of the Casimir energy density, it will roll down to the true minimum at $\tau_1 = \pm 1/2, \tau_2 = \sqrt{3}/2$ even with minimal amount of perturbations. This is shown in figure 4, 5. Observe that it tends to roll along the trail $\tau = 1$ in the moduli space.

When the tossing initial conditions are at a distant away from the saddle point and the true minimum, certain sets of the initial conditions still result in the stabilization of the torus moduli, τ_1, τ_2 , and the scale factor, b , of the extra dimension as is shown in figure 6, 7. Naturally, as long as the Casimir energy density at the stabilized value is

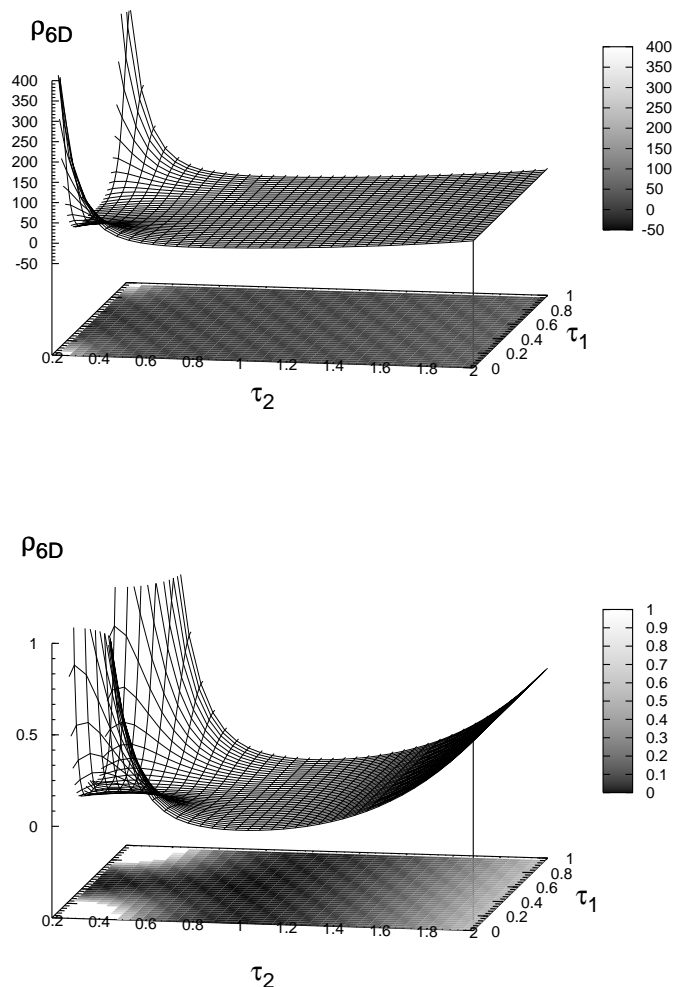


Figure 3: The Casimir energy density in six dimension from small bM approximation in the upper figure in comparison to the full expression in the lower figure. Both are evaluated at their corresponding b_{\min} .

positive, the acceleration of the scale factor, a , of the 4-dimensional spacetime is guaranteed. The positive Casimir density serves as the positive cosmological constant.

A natural consequence of the Casimir energy that is independent of the scale factor, $a(t)$, of the large dimension is the fact that it leads to $w_a = -1$ for the pressure $p_a = w_a \rho$. For the pressure in the compact extra dimensions, we can start by considering $p_b = -\partial(\rho V_b)/\partial V_b = w_b \rho$, w_b of our Casimir energy density is then given by

$$w_b = -1 - \frac{b}{2\rho} \frac{\partial \rho}{\partial b} \tag{5.1}$$

where ρ is the total Casimir energy density. Due to the dynamics of shape moduli (or

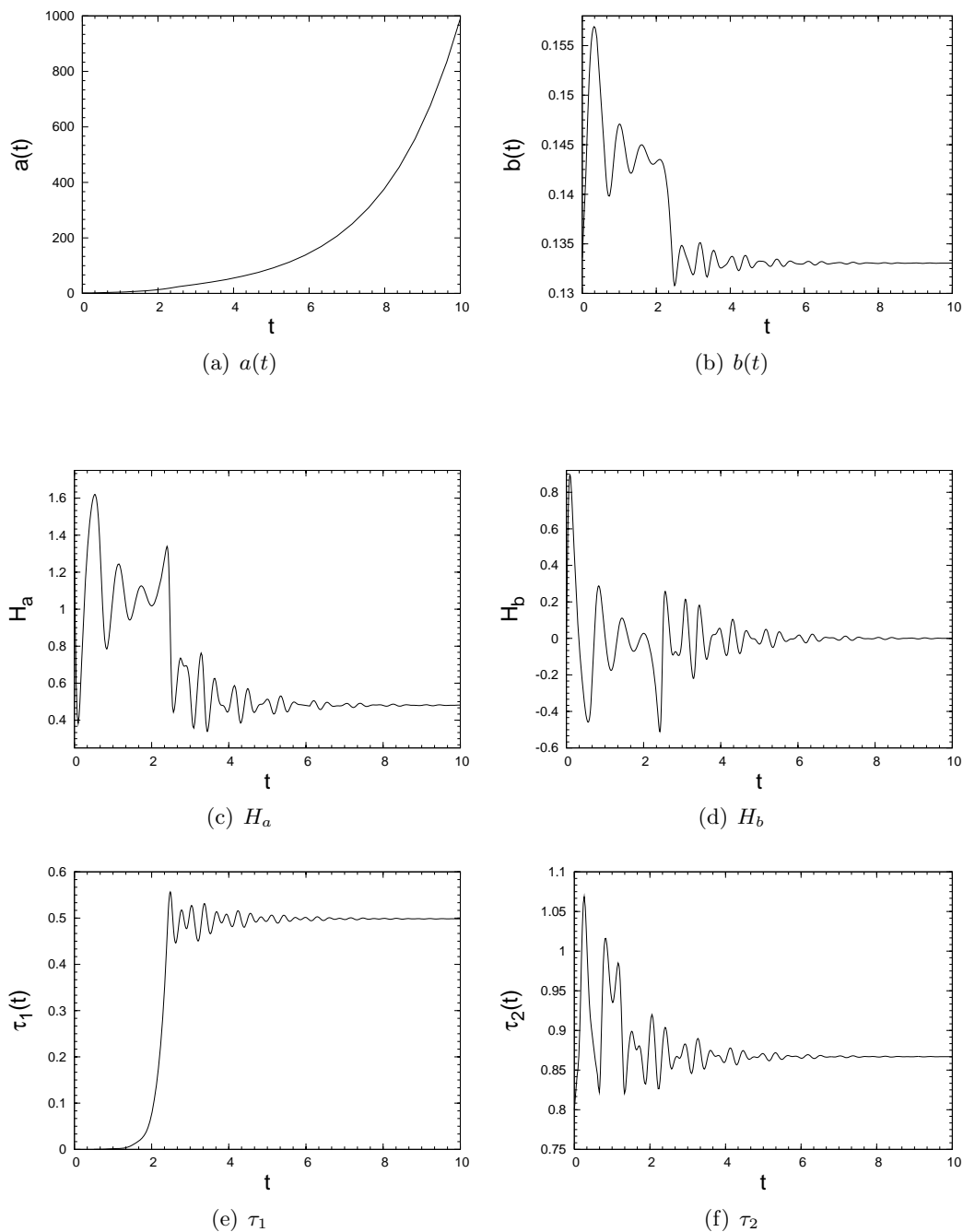


Figure 4: Cosmological dynamics when the universe is initially tossed very close to the saddle point $\tau_1 = 0, \tau_2 = 1$, it rolls along the trail $\tau = 1$ to the true minimum at $\tau_1 = \pm 1/2, \tau_2 = \sqrt{3}/2$.

Casimir “viscosity” in the compact space, see appendix B), the value of w_b at the stabilized radius at the true minimum is fractionally smaller than -2 (around -2.16) as is shown in figure 5.

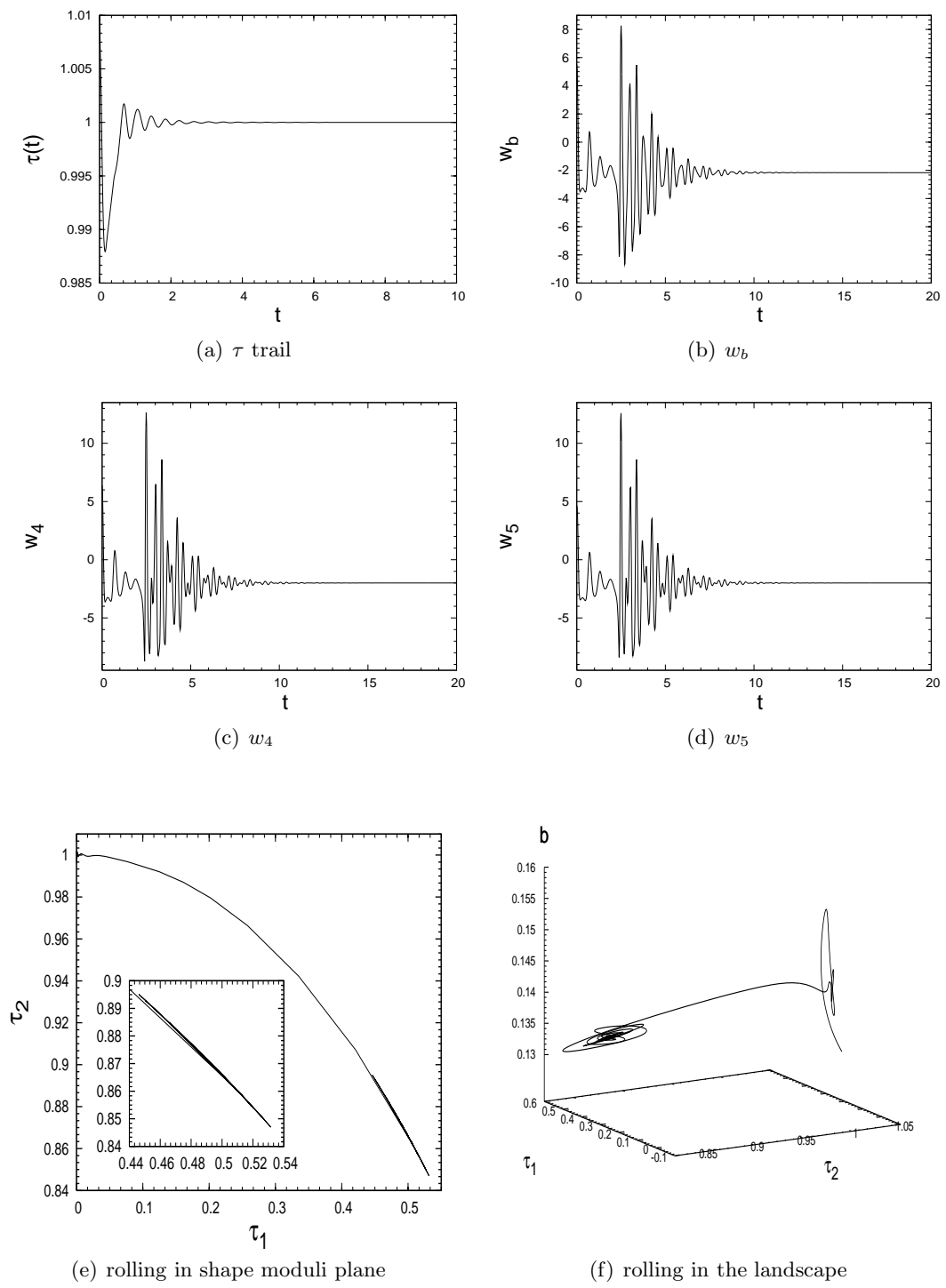


Figure 5: Rolling dynamics from saddle point to the true minimum.

A more appropriate definition of physical pressures in the distorted torus is

$$p_K^* \equiv T_K^K, \tag{5.2}$$

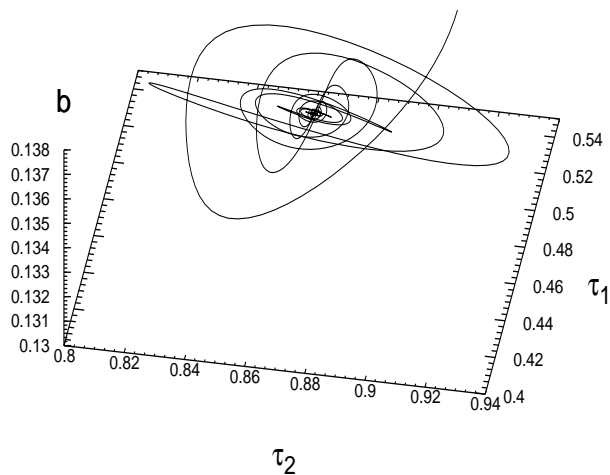


Figure 6: Rolling dynamics from other initial condition I.

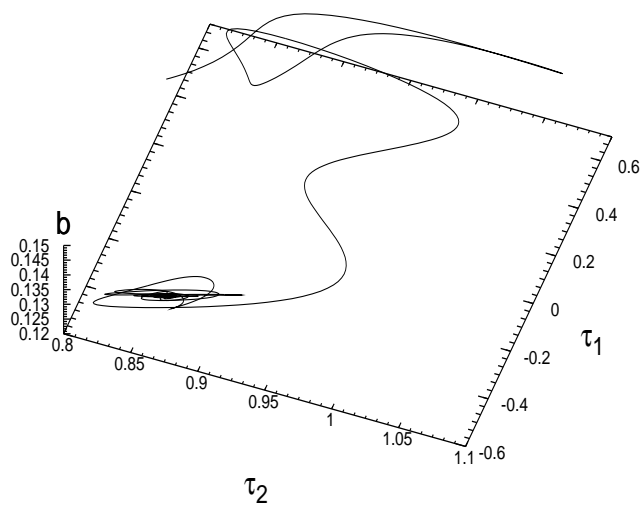


Figure 7: Rolling dynamics from other initial condition II.

where $K = 4, 5$. This definition gives the following expressions for $w_K = p_K^*/\rho$,

$$w_4 = -1 + \frac{b}{2\rho} \partial_b \rho \left(\frac{\tau_1^2 - \tau_2^2}{\tau_2^2} \right) + \frac{2\tau_1}{\rho} \partial_{\tau_1} \rho + \frac{1}{\rho} \left(\frac{\tau_2^2 - \tau_1^2}{\tau_2} \right) \partial_{\tau_2} \rho \quad (5.3)$$

$$w_5 = -1 + \frac{b}{2\rho} \partial_b \rho \left(\frac{\tau_1^2 - \tau_2^2}{\tau_2^2} \right) - \frac{1}{\rho} \left(\frac{\tau_2^2}{\tau_2} \right) \partial_{\tau_2} \rho. \quad (5.4)$$

By directly solving the equations of motion in six dimensions at the stabilized point where $\dot{H}_a = \dot{H}_b = H_b = \dot{\tau}_1 = \dot{\tau}_2 = 0$, it can be shown that $w_{4,5} = -2$, as is confirmed numerically in figure 5. It is interesting to note that the value of $w_{4,5}$ becomes -2 at both the saddle point and the true minimum where the dynamics is stabilized.

The difference of the two definitions of pressure originates from the *shear viscosity* induced by the Casimir energy in the off-diagonal components of the stress tensor. From the equations of motion of the 6-D universe with viscosities, eq. (B.10) in appendix B, shear viscosity at the stabilized point η_b^{stab} can be identified to be

$$\eta_b^{\text{stab}} = \frac{3H_{a,\text{stab}}}{16\pi G} \quad (5.5)$$

$$= \frac{\rho_{6D,\text{min}}}{2H_{a,\text{stab}}} \quad (5.6)$$

where $H_{a,\text{stab}}$ is the Hubble constant of the expanding four dimensions at the stabilized point of the compactified space. Note that we can evaluate eq. (2.3), (B.8) and (B.9) at the stabilized point and use the definition of η_b^{stab} to analytically confirm the numerical results in which $w_{4,5} = -2$ at the stabilized point.

We should mention here that the time scale, t_s , of the simulated figures is given by

$$t_s = \frac{\sqrt{23}}{2\pi} \frac{m_{pl}}{b_{\text{min}}} b_s^3, \quad (5.7)$$

where b_s is the scale of b , and $b_{\text{min}} \simeq 0.1328b_s$ as a result of numerical simulation. If we require that the stabilization time $\simeq 10t_s$ is less than the age of the universe, 10^{10} years, this will put constraint on the size b_{min} of the extra dimensions \mathbb{T}^2 ,

$$b_{\text{min}} \lesssim 0.7\mu\text{m}. \quad (5.8)$$

This is about few hundred times stronger than the constraints from table-top experiments [15].

It is interesting that in this kind of cosmological model, the constancy of the 4-dimensional gravitational constant, $G_4 = G/4\pi^2 b^2 = 1/m_{pl}^2$, up to the early times of the universe will give a very strong constraint on the size of the compactified extra dimensions. Any future observations of the universe from very early epoch could possibly put constraints on the inconstancy of the gravitational constant. Such constraints will put very strong limits on the size of compact extra dimensions in this kind of model where oscillatory behaviour is significant in the early times.

Another important aspect of this model is the relationship between the effective cosmological constant in 4-dimensional spacetime, $\Lambda_4 = 8\pi G_4 \rho_{4D,\text{min}}$, and the size of extra dimension, b_{min} ,

$$\Lambda_4 = 8\pi G_4 \rho_{4D,\text{min}} \quad (5.9)$$

$$= 3H_{E,\text{stab}}^2 \quad (5.10)$$

This leads to the typical value of $b_{\text{min}} \approx 2.4 \mu\text{m}$ for $\rho_{\text{vac}} \approx 10^{-11} \text{eV}^4$. The value of the effective size of extra dimensions, $2\pi b_{\text{min}} \approx 15\mu\text{m}$, yields the quantum gravity scale in the bulk, $M_* \approx 12 \text{TeV}$.

6. Conclusions and discussion

The stabilization of compact extra dimensions and the acceleration of the other 4-dimensional part of the spacetime can be simultaneously described by the dynamics of the Einstein field equations in the bulk spacetime. The acceleration of the 4-dimensional “universe” occurs naturally once the scale of the compact dimensions is stabilized and the density of the Casimir energy in the bulk becomes a (positive) constant at that stabilized value. As a result, the apparent positive “cosmological constant” that we seem to observe in the four dimensional visible universe is effectively induced. This is demonstrated beautifully in the work by Greene and Levin [3] when the Casimir density of the undistorted torus satisfies $w_a = -1, w_b = -2$ condition.

Shape moduli of the torus can be added to the model. The true minimum of the Casimir energy density of the torus with shape moduli is demonstrated to be located at $\tau_1 = \pm 1/2, \tau_2 = \sqrt{3}/2$. The cosmological dynamic shows that a minimally small perturbation to the saddle point rolls the universe down to the true minimum. Other initial conditions also suggest that the universe tends to roll around $\tau = 1$ contour to reach the true minimum. Note that it is also possible to stabilize the moduli at the saddle point $\tau_1 = 0, \tau_2 = 1$ but the initial conditions of the shape moduli fields must be fine-tuned so that $\tau_1 = 0, \dot{\tau}_1 = 0$. Some extra-mechanisms such as Brandenberger-Vafa mechanism in string gas cosmology [16] is needed for this purpose. However, as it was pointed out in [17], the stabilized point $\tau_1 = \pm 1/2$ and $\tau_2 = \sqrt{3}/2$ is also the fixed point of T-duality and the the enhance symmetry point hence Brandenberger-Vafa mechanism could also set the initial value of the moduli precisely to be at the stabilized point.

The shear viscosity in the extra dimension is determined to be proportional to the Hubble constant at the stabilized point, $\eta_b = 3H_{a,stab}/16\pi G$. Through the Einstein field equations, this Hubble constant of the 4-D universe is determined by the value of the Casimir energy density at the stabilized point. The effective four dimensional cosmological constant is also given by $8\pi G\rho_{6D,min}$.

In this kind of model, there is a relationship between the size of the compact dimensions and the observed four dimensional cosmological constant. This remarkable connection is induced by the nature of Casimir energy density which depends on the size of the compact dimension, resulting in $\Lambda_4 \sim b_{min}^{-6}$.

It is equally important to note that the constancy of the 4-D gravitational constant up to very early time of the universe will provide strong constraint on the size of extra dimension in this particular cosmological model which expresses oscillatory behaviour at the early times.

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A. Useful formulae

Phase space integration.

$$\int d^d k f(k) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int k^{d-1} f(k) dk. \quad (\text{A.1})$$

Poisson resummation.

$$\sum_{n \in \mathbb{Z}} f(n) = \tilde{f}(k) = \sqrt{2\pi} \sum_{m \in \mathbb{Z}} \tilde{f}(2\pi m), \quad (\text{A.2})$$

where

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx. \quad (\text{A.3})$$

If $f(x) = e^{-a(x+c)^2}$, then $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{a}} e^{-\frac{k^2}{4a} + ikc}$.

Integral representation of Gamma function.

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt. \quad (\text{A.4})$$

The integral representation of the modified Bessel function of the second kind.

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^{+\infty} t^{-\nu-1} e^{-t-\frac{z^2}{4t}} dt, \quad (\text{A.5})$$

where $|\arg(z)| < \frac{\pi}{2}$, $\text{Re}(z^2) > 0$.

Mellin transform.

$$z^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt e^{-zt} t^{s-1}; \quad \text{Re}(z) > 0, \quad \text{Re}(s) > 0. \quad (\text{A.6})$$

B. Energy momentum tensor of viscous fluid

Let $U^A = (1, 0, 0, 0, 0)$ be the 6-velocity of the cosmic fluid in comoving coordinates. In terms of the projection tensor $h_{AB} = g_{AB} + U_A U_B$, the general energy momentum tensor of fluid with bulk viscosity ζ and shear viscosity η is given by:

$$T_{AB} = \rho U_A U_B + (p - \zeta \theta) h_{AB} - 2\eta \sigma_{AB}. \quad (\text{B.1})$$

Here $\theta \equiv \nabla_A U^A$ is the scalar expansion and $\sigma_{AB} = h_A^C h_B^D \nabla_{(C} U_{D)} - \frac{1}{5} h_{AB} \theta$ is the shear tensor. By using metric defined in eq. (2.1) and (2.2), we can show that

$$T_0^0 = -\rho, \quad (\text{B.2})$$

$$T_1^1 = T_2^2 = T_3^3 = p_a \quad (\text{B.3})$$

$$T_4^4 = (p_b - \zeta_b \theta) - 2\eta_b \left[\frac{3}{5}(H_b - H_a) - \frac{\dot{\tau}_2}{2\tau_2} - \frac{\tau_1 \dot{\tau}_1}{2\tau_2^2} \right] \quad (\text{B.4})$$

$$T_5^5 = (p_b - \zeta_b \theta) - 2\eta_b \left[\frac{3}{5}(H_b - H_a) + \frac{\dot{\tau}_2}{2\tau_2} + \frac{\tau_1 \dot{\tau}_1}{2\tau_2^2} \right] \quad (\text{B.5})$$

$$T_5^4 = 2\eta_b \left[\frac{\tau_1 \dot{\tau}_2}{\tau_2} + (\tau_1^2 - \tau_2^2) \frac{\dot{\tau}_1}{2\tau_2^2} \right] \quad (\text{B.6})$$

$$T_4^5 = -\eta_b \frac{\dot{\tau}_1}{\tau_2^2} \quad (\text{B.7})$$

Here we assume there is no viscosity in noncompact large dimensions ($\zeta_a = \eta_a = 0$). Einstein's equations, eq. (2.4)–(2.7), can be written in terms of bulk and shear viscosity as

$$\dot{H}_a + 3H_a^2 + 2H_a H_b = \frac{8\pi G}{4} \left\{ \rho_{6D} + p_a - 2(p_b - \zeta_b \theta) + \frac{12}{5} \eta_b (H_b - H_a) \right\}, \quad (\text{B.8})$$

$$\dot{H}_b + 2H_b^2 + 3H_a H_b = \frac{8\pi G}{4} \left\{ \rho_{6D} - 3p_a + 2(p_b - \zeta_b \theta) - \frac{12}{5} \eta_b (H_b - H_a) \right\}, \quad (\text{B.9})$$

$$\ddot{\tau}_1 + \left(3H_a + 2H_b - 2\frac{\dot{\tau}_2}{\tau_2} \right) \dot{\tau}_1 = 16\pi G \{ \eta_b \dot{\tau}_1 \}, \quad (\text{B.10})$$

$$\frac{\ddot{\tau}_2}{\tau_2} + \frac{\dot{\tau}_1^2 - \dot{\tau}_2^2}{\tau_2^2} + 3H_a \frac{\dot{b}}{\tau_2} + 2H_b \frac{\dot{\tau}_2}{\tau_2} = 48\pi G \left\{ \eta_b \frac{\dot{\tau}_2}{\tau_2} \right\}. \quad (\text{B.11})$$

The conservation of energy is

$$\begin{aligned} & \dot{\rho}_{6D} + 3H_a(\rho_{6D} + p_a) + 2H_b(\rho_{6D} + p_b) \\ & + \left(\frac{12}{5} \eta_b - 6\zeta_b \right) H_a H_b - \left(\frac{12}{5} \eta_b + 4\zeta_b \right) H_b^2 - \eta_b \left(\frac{\dot{\tau}_1^2}{\tau_2^2} + \frac{\dot{\tau}_2^2}{\tau_2^2} \right) = 0. \end{aligned} \quad (\text{B.12})$$

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